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Nonlinear Schrödinger Dynamics and Nonlinear Observables

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Abstract

It is is explained why physical consistency requires substituting linear observables by nonlinear ones for quantum systems with nonlinear time evolution of pure states. The exact meaning and the concrete physical interpretation are described in full detail for a special case of the nonlinear Doebner-Goldin equation.

1 Schrödinger dynamics

By *Schrödinger dynamics* we mean a strongly continuous two-parameter family of mappings $\hat{\beta}_{t_2,t_1}$ of some Hilbert space $\mathcal{H}$ of (pure) states onto itself, defined for $t_1, t_2 \in \mathbb{R}$ and satisfying the following conditions:

$$
\begin{aligned}
\hat{\beta}_{t,t}(\Psi) &= \Psi && \forall\, t \in \mathbb{R}\,,\ \Psi \in \mathcal{H}\,, && (1)\\
\hat{\beta}_{t_3,t_2}\Big(\hat{\beta}_{t_2,t_1}(\Psi)\Big) &= \hat{\beta}_{t_3,t_1}(\Psi) && \forall\, t_1, t_2, t_3 \in \mathbb{R}\,,\ \Psi \in \mathcal{H} \setminus \{0\}\ , && (2)\\
\hat{\beta}_{t_2,t_1}(c\Psi) &= c\hat{\beta}_{t_2,t_1}(\Psi) && \forall\, t_1, t_2 \in \mathbb{R}\,,\ c \in \mathbb{C}\,,\ \Psi \in \mathcal{H} \setminus \{0\}\ . && (3)
\end{aligned}
$$

In order to be consistent with the standard (nonrelativistic) interpretation

$$
\begin{aligned}
\rho_{\Psi_t} &\overset{\text{def}}{=} |\Psi_t|^2 \\
&= \text{probability density for particle position at time } t\,,
\end{aligned} \tag{4}
$$

we add the requirement

$$\left\|\hat{\beta}_{t_2,t_1}(\Psi)\right\| = \|\Psi\| \quad \forall\, t_1, t_2 \in \mathbb{R}\ \Psi \in \mathcal{H}\,. \tag{5}$$

In other words:[1]

> A Schrödinger dynamics is a norm conserving propagator fulfilling (3).

We call a Schrödinger dynamics $\left\{\hat{\beta}_{t_2,t_1}\right\}$ *linear* if

$$\left|\left\langle \frac{\hat{\beta}_{t_2,t_1}(\Phi)}{\left\|\hat{\beta}_{t_2,t_1}(\Phi)\right\|}\middle|\frac{\hat{\beta}_{t_2,t_1}(\Psi)}{\left\|\hat{\beta}_{t_2,t_1}(\Psi)\right\|}\right\rangle\right| = \left|\left\langle \frac{\Phi}{\|\Phi\|}\middle|\frac{\Psi}{\|\Psi\|}\right\rangle\right| \quad \forall\, t_1, t_2 \in \mathbb{R}\,,\ \Psi, \Phi \in \mathcal{H} \setminus \{0\} \tag{6}$$

holds. Otherwise it is called *nonlinear*. Actually, since (1) and (2) imply

$$\hat{\beta}_{t_1,t_2} = \hat{\beta}^{-1}_{t_2,t_1} \quad \forall\, t_1, t_2 \in \mathbb{R}\,,$$

a Schrödinger dynamics is given by the one-parameter family of invertible norm conserving mappings $\beta_t \stackrel{\text{def}}{=} \hat{\beta}_{t,0}$:

$$\hat{\beta}_{t_2,t_1} = \hat{\beta}_{t_2,0} \circ \hat{\beta}^{-1}_{t_1,0} \quad \forall\, t_1, t_2 \in \mathbb{R}\,.$$

Typically, such a family is fixed by some nonlinear Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\beta_t(\Psi) = \boldsymbol{H}_t\Big(\beta_t(\Psi)\Big)\,, \tag{7}$$

with a suitable nonlinear Hamiltonian $\boldsymbol{H}_t$ fulfilling

$$\boldsymbol{H}_t(c\Psi) = c\boldsymbol{H}_t(\Psi) \quad \forall\, c \in \mathbb{C} \tag{8}$$

on a suitable dense set of state vectors. For example,

$$H_t(\Psi) = \left(-\frac{\hbar^2}{2m}\Delta + V(\vec{x},t)\right)\Psi + iD\hbar G(\Psi)\,, \quad G(\Psi) \stackrel{\text{def}}{=} \vec{\nabla}^2\Psi + \left|\frac{\vec{\nabla}\Psi}{\Psi}\right|^2\Psi \tag{9}$$

was considered in [3] and generalized[2] to

$$\begin{aligned} H_t(\Psi) \;=\; & \left(-\tfrac{\hbar^2}{2m}\Delta + V(\vec{x},t)\right)\Psi + i\tfrac{\hbar D}{2}\tfrac{\Delta\rho_\Psi}{\rho_\Psi}\Psi \\ & +\hbar D\left(c_1\tfrac{\vec{\nabla}\cdot\vec{J}_\Psi}{\rho_\Psi} + c_2\tfrac{\Delta\rho_\Psi}{\rho_\Psi} + c_3\tfrac{\vec{J}_\Psi^2}{\rho_\Psi^2} + c_4\tfrac{\vec{J}_\Psi\cdot\vec{\nabla}\rho_\Psi}{\rho_\Psi^2} + c_5\tfrac{(\vec{\nabla}\rho_\Psi)^2}{\rho_\Psi^2}\right)\Psi \end{aligned} \tag{10}$$

---

[1] In order to allow, e.g., for the Bialynicki-Birula equation [1], condition (3) could be generalized to

$$\hat{\beta}_{t_2,t_1}(c\Psi) = c\, e^{i\varphi(c,t_1,t_2)}\,\hat{\beta}_{t_2,t_1}(\Psi) \quad \forall\, t_1, t_2 \in \mathbb{R}\,,\ c \in \mathbb{C}\,,\ \Psi \in \mathcal{H} \setminus \{0\}\,,$$

where φ is some real-valued function. Since we are mainly interested in the Doebner-Goldin equation, here, this generalization is not necessary. The physical importance of (3) was extensively discussed in [8].

[2] Equation (9) is the special case $c_1 = -1\,,\ c_2 = \ldots = c_5 = 0$ of (10).

where

$$\rho_\Psi \stackrel{\text{def}}{=} |\psi|^2 \ , \quad \vec{J}_\Psi \stackrel{\text{def}}{=} \frac{1}{2i}\left(\overline{\Psi}\vec{\nabla}\Psi - \Psi\vec{\nabla}\overline{\Psi}\right)$$

[4]. For

$$c_1 = 1 \ , \quad c_2 + 2c_5 = 0 \ , \quad c_3 = 0 \ , \quad c_4 = -1 \ , \tag{11}$$

the linear dynamics $\beta_{0,t}$, characterized by[3]

$$i\hbar\partial_t \, \beta_{0,t}\Psi = \boldsymbol{H}_{0,t}\, \beta_{0,t}\Psi \ , \quad \boldsymbol{H}_{0,t} = -\frac{\hbar^2}{2m}\Delta + V(\vec{x},t) \ , \tag{12}$$

is *affiliated* to β_t by some nonlinear norm conserving intertwining operator $\boldsymbol{N}$:

$$\beta_t \circ \boldsymbol{N} = \boldsymbol{N} \circ \beta_{0,t}$$

(see [10]). Let us restrict to the case $c_2 = -\frac{mD}{\hbar}$. Then this intertwiner is given by $\boldsymbol{N} = \boldsymbol{N}_D$, where

$$\left(\boldsymbol{N}_D(\Psi)\right)(\vec{x}) \stackrel{\text{def}}{=} \begin{cases} e^{i\frac{mD}{\hbar}\ln\rho_\Psi(\vec{x})}\Psi(\vec{x}) & \text{if } \Psi(\vec{x}) \text{ defined and} \neq 0 \\ 0 & \text{else} \end{cases} . \tag{13}$$

(see [9, Sect. 3.4]). By Lebesque's bounded convergence theorem [6, p. 110] it is easily seen that (13) defines a strongly continuous norm conserving mapping $\boldsymbol{N}_D$ from $\mathcal{H}$ onto itself with inverse

$$\left(\boldsymbol{N}_D\right)^{-1} = \boldsymbol{N}_{-D} \, . \tag{14}$$

Actually (13), (7) and[4]

$$\beta_t = \beta_{D,t} \stackrel{\text{def}}{=} \boldsymbol{N}_D \circ \beta_{0,t} \circ \boldsymbol{N}_{-D} \tag{15}$$

should be taken as a definition for (10) when $c_1 = -c_4 = 1$ and $c_2 = c_3 = c_5 = 0$.

2 Consequences of nonlinearity

Let us denote by $P(\mathcal{H})$ the set of all orthogonal projections in $\mathcal{H}$:

$$P(\mathcal{H}) \stackrel{\text{def}}{=} \{\boldsymbol{P} \in \mathcal{B}(\mathcal{H}) : \boldsymbol{P} = \boldsymbol{P}^*\boldsymbol{P}\} \ .$$

Then the following is well known (see, e.g., [12, §3–2]).

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[3] For time-independent V we have $\beta_{0,t} = e^{-\frac{i}{\hbar}\boldsymbol{H}_{0,0}t}$, of course.

[4] Note, however, that

$$\boldsymbol{H}_{D,t} \stackrel{\text{def}}{=} \left(i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\beta_{D,t}^{-1}\right) \circ \beta_{D,t} \neq \boldsymbol{N}_D \circ \boldsymbol{H}_t^0 \circ \boldsymbol{N}_{-D} \, ,$$

as can be easily checked when $\Psi(\vec{x}) = e^{-x}$ in some open region, even though

$$\langle \Psi \mid \boldsymbol{H}_{D,t}(\Psi)\rangle = \left\langle \boldsymbol{N}_{-D}(\Psi) \mid \boldsymbol{H}_t^0 \boldsymbol{N}_{-D}(\Psi)\right\rangle \, .$$

What about the relation between $\|\boldsymbol{H}_{D,t}(\Psi)\|$ and $\|\boldsymbol{H}_0\boldsymbol{N}_{-D}(\Psi)\|$?

Theorem 2.1 (Wigner) *A Schrödinger dynamics* $\{\beta_t\}_{t\in\mathbb{R}}$ *on* $\mathcal{H}$, *where* $\dim\mathcal{H}\geq 3$, *is linear if and only if for every* $t\in\mathbb{R}$ *there is either a unitary or an anti-unitary*[5] *operator* $\boldsymbol{U}_t$ *with*

$$\boldsymbol{P}_{\beta_t(\Psi)}=\boldsymbol{P}_{\boldsymbol{U}_t\Psi}=\boldsymbol{U}_t\boldsymbol{P}_\Psi\boldsymbol{U}_t^*\quad\forall\,\Psi\in\mathcal{H}\,,$$

where

$$\boldsymbol{P}_\Psi\,\Phi\stackrel{\mathrm{def}}{=}\|\Psi\|^{-2}\,\langle\Psi\mid\Phi\rangle\,\Psi\quad\textit{for }\Phi\in\mathcal{H}\,,\ \Psi\in\mathcal{H}\setminus\{0\}\ .$$

Perhaps less well-known is the following.

Theorem 2.2 *A Schrödinger dynamics* $\{\beta_t\}_{t\in\mathbb{R}}$ *is linear if and only the following statement is correct:*
Let

$$\sum_{\nu=0}^{\infty}\underbrace{\lambda_\nu}_{\geq 0}=\sum_{\nu'=0}^{\infty}\underbrace{\lambda'_{\nu'}}_{\geq 0}=1\,;\qquad\{\Psi_\nu\}\,,\{\Psi'_{\nu'}\}\subset\mathcal{H}\setminus\{0\}\ .$$

Then

$$\sum_{\nu=0}^{\infty}\lambda_\nu\omega_{\Psi_\nu}(\boldsymbol{P})=\sum_{\nu'=0}^{\infty}\lambda'_{\nu'}\omega_{\Psi'_{\nu'}}(\boldsymbol{P})\Longrightarrow\sum_{\nu=0}^{\infty}\lambda_\nu\omega_{\beta_t(\Psi_\nu)}(\boldsymbol{P})=\sum_{\nu'=0}^{\infty}\lambda'_{\nu'}\omega_{\beta_t(\Psi'_{\nu'})}(\boldsymbol{P})\tag{16}$$

holds for all $t\in\mathbb{R}$ *and* $\boldsymbol{P}\in P(\mathcal{H})$ *,where*

$$\omega_\Psi(\boldsymbol{P})\stackrel{\mathrm{def}}{=}\frac{\langle\Psi\mid\boldsymbol{P}\Psi\rangle}{\langle\Psi\mid\Psi\rangle}\ \textit{for }\Psi\in\mathcal{H}\setminus\{0\}\ ,\ \boldsymbol{P}\in P(\mathcal{H})\,.$$

Proof of Theorem 2.2: As usual, denote by $\mathcal{T}(\mathcal{H})$ the set of trace class operators on $\mathcal{H}$. Assume (16). Then

$$\begin{gathered}\alpha_t(\boldsymbol{T})\stackrel{\mathrm{def}}{=}\textstyle\sum_{\nu=0}^{\infty}\lambda_\nu\boldsymbol{P}_{\beta_t(\Psi_\nu)}\\ \text{for }\boldsymbol{T}=\textstyle\sum_{\nu=0}^{\infty}\lambda_\nu\boldsymbol{P}_{\Psi_\nu}\in S(\mathcal{H})\stackrel{\mathrm{def}}{=}\{\boldsymbol{T}\in\mathcal{T}(\mathcal{H}):\,\boldsymbol{T}\geq 0\,,\ \mathrm{tr}(\boldsymbol{T})=1\}\end{gathered}$$

is a consistent definition. For $\boldsymbol{T}\in\mathcal{T}(\mathcal{H})$ let us denote by $\pm\boldsymbol{T}_\pm$ its positive resp. negative part:

$$\boldsymbol{T}=\boldsymbol{T}_+-\boldsymbol{T}_-\,;\quad\boldsymbol{T}_+,\boldsymbol{T}_-\geq\mathbf{0}\,.$$

Then the consistent extension

$$\alpha_t(\boldsymbol{T})\stackrel{\mathrm{def}}{=}\mathrm{tr}(\boldsymbol{T}_+)\,\alpha_t\left(\frac{\boldsymbol{T}_+}{\mathrm{tr}(\boldsymbol{T}_+)}\right)-\mathrm{tr}(\boldsymbol{T}_-)\,\alpha_t\left(\frac{\boldsymbol{T}_-}{\mathrm{tr}(\boldsymbol{T}_-)}\right)\quad\text{for }t\in\mathbb{R}\,,\,\boldsymbol{T}\in\mathcal{T}(\mathcal{H})\,.$$

[5] Actually, thanks to strong continuity of β_t and $\beta_0=\mathbf{1}$, $\boldsymbol{U}_t$ cannot be anti-unitary. But this is of no relevance here.

defines a family of invertible linear mappings $\alpha_t : \mathcal{T}(\mathcal{H}) \longrightarrow \mathcal{T}(\mathcal{H})$, mapping $S(\mathcal{H})$ onto itself. Such mappings are known to be implemented by either unitary or anti-unitary operators U_T :

$$\boldsymbol{P}_{\beta_t(\Psi)} = \alpha_t(\boldsymbol{P}_\Psi) = \boldsymbol{U}_t \boldsymbol{P}_\Psi \boldsymbol{U}_t^* \quad \forall\, \Psi \in \mathcal{H} \setminus \{0\}$$

(see [2, Corollary 2.3.2]). By (the easy part of) Wigner's theorem this shows that $\{\beta_t\}$ is linear. Conversely, (16) follows from linearity of the Schrödinger dynamics by (the nontrivial part of) Wigner's theorem. ∎

Lemma 2.3 *A Schrödinger dynamics $\{\beta_t\}_{t\in\mathbb{R}}$ is linear if and only if one of the following (equivalent) statements is correct:*

(i) The following ***primitive causality*** *condition holds:*

Let $t \in \mathbb{R}$ and $\boldsymbol{P} \in P(\mathcal{H})$. Then there is a $\boldsymbol{P}' \in P(\mathcal{H})$ fulfilling

$$\omega_\Psi(\boldsymbol{P}) = \omega_{\beta_t(\Psi)}(\boldsymbol{P}') \quad \forall\, \Psi \in \mathcal{H}\,.$$

(ii) There is a ***Heisenberg picture*** *in the usual sense:*

Let $t \in \mathbb{R}$ and $\boldsymbol{P} \in P(\mathcal{H})$. Then there is a $\boldsymbol{P}(t) \in P(\mathcal{H})$ fulfilling

$$\omega_{\beta_t(\Psi)}(\boldsymbol{P}) = \omega_\Psi\Big(\boldsymbol{P}(t)\Big) \quad \forall\, \Psi \in \mathcal{H}\,.$$

(iii) Let $t \in \mathbb{R}$ and $\boldsymbol{P}, \boldsymbol{P}' \in P(\mathcal{H})$. Then

$$\omega_\Psi(\boldsymbol{P}) = 1 \iff \omega_{\beta_t(\Psi)}(\boldsymbol{P}') = 1 \quad \forall\, \Psi \in \mathcal{H}\,.$$

implies

$$\omega_\Psi(\boldsymbol{P}) = 0 \iff \omega_{\beta_t(\Psi)}(\boldsymbol{P}') = 0 \quad \forall\, \Psi \in \mathcal{H}\,.$$

3 Faster than light signals?

If the Schrödinger dynamics $\{\beta_t\}_{t\in\mathbb{R}}$ is nonlinear, then (16) does not hold for all $t \in \mathbb{R}$ and $\boldsymbol{P} \in P(\mathcal{H})$. This can be understood as a warning that causality problems (faster than light signals) might arise in a nonlinear theory.[6] Let us have a qualitative discussion of this problem without any assumption concerning the measuring process.

[6]See [5] and also Gisin's contribution to these proceedings.

To have a simple model let us begin with standard Schrödinger theory of one particle in an external potential $V(\vec{x},t)$, i.e. time-evolution is given by some unitary propagator $\boldsymbol{U}_t$ depending on $V(\vec{x},t)$:

$$i\hbar\partial_t \boldsymbol{U}_t\Phi = \left(-\frac{\hbar^2}{2m}\Delta_{\vec{x}} + V(\vec{x},t)\right)\boldsymbol{U}_t\Phi\,.$$

Now consider the nonlinear Schrödinger dynamics on $\mathcal{H} = L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^3)$ defined by (15), where

$$\beta_{0,t} = \boldsymbol{U}_t\otimes\boldsymbol{U}_t\,,$$

$\boldsymbol{N}_D$ being always given by (13) whatever the dimension of $\vec{x}$-space may be:

$$\boldsymbol{N}_D(\Psi) = e^{i\frac{mD}{\hbar}\ln\rho_\Psi}\Psi \quad \text{almost everywhere}\,.$$

Then for $\Phi_1,\Phi_2,\hat{\Phi}_1,\hat{\Phi}_2\in L^2(\mathbb{R}^3)$

$$\Psi_t \stackrel{\text{def}}{=} \boldsymbol{N}_D\left(\sum_j(\boldsymbol{U}_t\Phi_j)\otimes\left(\boldsymbol{U}_t\hat{\Phi}_j\right)\right)$$

is a solution of the corresponding nonlinear Schrödinger equation

$$\begin{aligned} i\hbar\partial_t\Psi_t = \;& \left(-\tfrac{\hbar^2}{2m}\Delta_{\vec{X}} + V(\vec{x}_1,t)+V(\vec{x}_2,t)\right)\Psi_t + i\tfrac{\hbar D}{2}\tfrac{\Delta_{\vec{X}}\rho_{\Psi_t}}{\rho_{\Psi_t}}\Psi_t \\ & +\hbar D\left(\tfrac{\vec{\nabla}_{\vec{X}}\cdot\vec{J}_{\Psi_t}}{\rho_{\Psi_t}} - \tfrac{mD}{\hbar}\tfrac{\Delta\rho_\Psi}{\rho_\Psi} - \tfrac{\vec{J}_{\Psi_t}\cdot\vec{\nabla}_{\vec{X}}\rho_{\Psi_t}}{\rho^2_{\Psi_t}} + \tfrac{mD}{2\hbar}\tfrac{(\vec{\nabla}\rho_\Psi)^2}{\rho^2_\Psi}\right)\Psi_t\,,\quad \vec{X}=(\vec{x}_1,\vec{x}_2)\,,\end{aligned}$$

for which we assume 4 as in ordinary two-particle Schrödinger theory. Even though $\boldsymbol{N}_D$ is nonlinear, we always have

$$\Psi_j\Psi_k = 0 \text{ for } j\neq k \implies \boldsymbol{N}_D\left(\sum_k c_k\Psi_k\right) = c_k\sum_k\boldsymbol{N}_D(\Psi_k)\,. \tag{17}$$

Moreover, we have the separability property

$$\boldsymbol{N}_D(\Phi\otimes\Psi) = \boldsymbol{N}_D(\Phi)\otimes\boldsymbol{N}_D(\Psi)\,. \tag{18}$$

Let us assume that the $\boldsymbol{U}_t\Phi_j$ are (essentially) localized in the 'laboratory' and that their supports are (essentially) disjoint. Then, by (17) and (18) we have

$$\Psi_t = \sum_j \boldsymbol{N}_D\left(\boldsymbol{U}_t\Phi_j\right)\otimes\boldsymbol{N}_D\left(\boldsymbol{U}_t\hat{\Phi}_j\right)$$

and, consequently,

$$\langle\Psi_t\mid(\boldsymbol{A}\otimes\mathbf{1})\Psi_t\rangle = \sum_{j,k}\lambda_{jk}(t)\,\langle\boldsymbol{N}_D(\boldsymbol{U}_t\Phi_j)\mid\boldsymbol{A}\,\boldsymbol{N}_D(\boldsymbol{U}_t\Phi_k)\rangle\;,$$

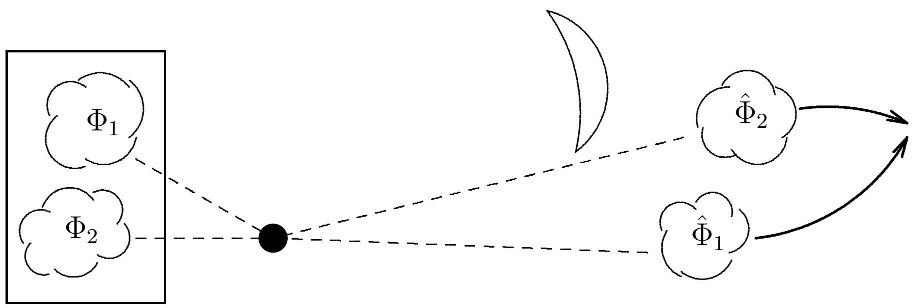
where

$$\lambda_{jk}(t) \stackrel{\text{def}}{=} \left\langle \boldsymbol{N}_D(\boldsymbol{U}_t\hat{\Phi}_j) \mid \boldsymbol{N}_D(\boldsymbol{U}_t\hat{\Phi}_k)\right\rangle \,,$$

is (essentially) valid. Therefore, as long as the vectors $\boldsymbol{N}_D(\boldsymbol{U}_t\hat{\Phi}_k)$ are pairwise orthogonal the partial state with respect to the 'observables' $\boldsymbol{A} \otimes \mathbf{1}$ is (essentially) the mixed state

$$\omega_{\Psi_t}(\boldsymbol{A} \otimes \mathbf{1}) = \sum_j \lambda_{jj}\, \omega_{\boldsymbol{N}_D(\boldsymbol{U}_t\Phi_k)}(\boldsymbol{A})\,. \tag{19}$$

Assume the supports of the $\boldsymbol{U}_t\hat{\Phi}_k$ to be 'behind the moon' and initially pairwise disjoint.[7] So the $\boldsymbol{N}_D(\boldsymbol{U}_t\hat{\Phi}_k)$ are initially pairwise orthogonal.[8] Thanks to nonlinearity of $\boldsymbol{N}_D$, however, this orthogonality can be (sufficiently) destroyed by applying a suitable exterior field 'behind the moon' causing the (essential) supports of the $\boldsymbol{U}_t\hat{\Phi}_k$ to overlap:[9]



This way the partial state with respect to the operators $\boldsymbol{A} \otimes \mathbf{1}$ changes in a way not depending on the distance of the 'moon' from the 'laboratory'.

> Clearly things can be arranged[10] such as to produce **faster than light signals** in a realistic way, **if all linear observables can really be measured**.

Admittedly, we used the additional assumption, that Ψ_t describes a state that can experimentally prepared.

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[7]Then (19) is exact.

[8]Note that $\operatorname{supp}\boldsymbol{N}_D(\Psi) \subset \Psi$.

[9]For instance, if (for $\vec{x} = x \in \mathbb{R}^1$) we define

$$\Phi_\pm(x) = \begin{cases} \pm e^{\pm\frac{\hbar\pi}{4mD}} & \text{for } x \in (0,+1)\,, \\ 1 & \text{for } x \in (-1,0)\,, \\ 0 & \text{else}\,, \end{cases}$$

we have

$$\langle \Phi_- \mid \Phi_+ \rangle = 0\,, \quad \frac{\langle \boldsymbol{N}_D(\Phi_-) \mid \boldsymbol{N}_D(\Phi_+)\rangle}{\|\Phi_-\|\;\|\Phi_+\|} = \sqrt{\frac{2}{1+\cosh\left(\frac{\hbar\pi}{2mD}\right)}} \;\left(\approx 0.755 \text{ for } \frac{\hbar}{mD} = 1\right).$$

[10]In principle these considerations can be made numerically precise.

4 Generalized projection valued measures

The essential message of Theorem 2.2, Lemma 2.3 and the discussion in Section 3 is twofold:

(i) For a nonlinear theory **not all** $\boldsymbol{P} \in P(\mathcal{H})$ should be considered as actually **measurable** (in principle).

(ii) For a nonlinear theory one should **add** some kind of **nonlinear observables** to identify the initial conditions of classical mixtures and to restore primitive causality as well as the Heisenberg picture.

Definition 4.1 *A one-parameter family* $\boldsymbol{A}$ *of mappings* $\boldsymbol{E}_B : \mathcal{H} \longrightarrow \mathcal{H}$*, defined for all Borel sets* $B \subset \mathrm{I\!R}$ *is called a **generalized projection valued measure***[11] (GPVM)*, if the following requirements are fulfilled:*

(i) For every $\Psi \in \mathcal{H} \setminus \{0\}$

$$B \longmapsto \mu_\psi^{\boldsymbol{A}}(B) \stackrel{\text{def}}{=} \omega_\Psi(\boldsymbol{E}_B) = \frac{\|\boldsymbol{E}_B(\Psi)\|^2}{\|\Psi\|^2}$$

defines a probability measure $\mu_\psi^{\boldsymbol{A}}$ *on* $\mathrm{I\!R}$.

(ii) For every pair of Borel sets $B_1, B_2 \subset \mathrm{I\!R}$:

$$\boldsymbol{E}_{B_1} \circ \boldsymbol{E}_{B_2} = \boldsymbol{E}_{B_1 \cap B_2} \,.$$

(iii) For every $\Psi \in \mathcal{H} \setminus \{0\}$ *and for every Borel set* $B \in \mathrm{I\!R}$:

$$\mu_\psi^{\boldsymbol{A}}(B) = 1 \implies \boldsymbol{E}_B(\Psi) = \Psi \,.$$

Definition 4.2 *A* GPVM $\boldsymbol{A} = \{\boldsymbol{E}_B\}$ *is called an **observable** if for every Borel set* $B \subset \mathrm{I\!R}$ *there is (in principle) an experimental test with the following properties:*

(i) The probability to get a positive answer if the quantum system is in the state ω_Ψ *is* $\mu_\Psi^{\boldsymbol{A}}(B)$.

(ii) The effect of such a test on the quantum system is described by the instantaneous change

$$\Psi \longmapsto \boldsymbol{E}_B(\Psi) \qquad \textit{(wave packet collapse à la Lüders)}$$

of the state vector Ψ .

[11]Maybe this is not a good terminology, because it is only the probabilities that are additive.

An observable $\boldsymbol{A} = \{\boldsymbol{E}_B\}$ is called ***bounded****, if $\boldsymbol{E}_C = \mathbb{1}$ for some compact subset C of $\mathbb{R}$. It is called* ***linear*** *if all the $\boldsymbol{E}_B$ are orthogonal projections. The* ***expectation value*** *for an observable $\boldsymbol{A} = \{\boldsymbol{E}_B\}$ in the state characterized by $\Psi \in \mathcal{H}$ is*

$$E_\Psi(\boldsymbol{A}) \stackrel{\text{def}}{=} \int \lambda \, \mathrm{d}\mu_\psi^{\boldsymbol{A}}(\lambda) \, .$$

The observable is called ***conserved*** *by the Schrödinger dynamics β_t if $\boldsymbol{E}_B \circ \beta_t = \beta_t \circ \boldsymbol{E}_B$ holds for all $t \in \mathbb{R}$ and for all Borel sets $B \subset \mathbb{R}$.*

Let the quantum system be in the state ω_Ψ at time $t = 0$. Then the probability for getting a positive outcome for both a test at time $t_1 > 0$ corresponding to $\boldsymbol{E}_{B_1}^{\boldsymbol{A}_1}$ and a test at time $t_2 > t_1$ corresponding to $\boldsymbol{E}_{B_2}^{\boldsymbol{A}_2}$ is

$$\begin{aligned}
\mu^{\boldsymbol{A}_1}_{\beta_{t_1}(\Psi)}(B_1)\,\mu^{\boldsymbol{A}_2}_{(\hat\beta_{t_2,t_1}\circ \boldsymbol{E}^{\boldsymbol{A}_1}_{B_1}\circ\beta_{t_1})(\Psi)}(B_2) \quad &= \quad \frac{\left\|\boldsymbol{E}^{\boldsymbol{A}_1}_{B_1}\big(\beta_{t_1}(\Psi)\big)\right\|^2}{\left\|\beta_{t_1}(\Psi)\right\|^2}\;\frac{\left\|\left(\boldsymbol{E}^{\boldsymbol{A}_2}_{B_2}\circ\hat\beta_{t_2,t_1}\circ\boldsymbol{E}^{\boldsymbol{A}_1}_{B_1}\circ\beta_{t_1}\right)(\Psi)\right\|^2}{\left\|\left(\hat\beta_{t_2,t_1}\circ\boldsymbol{E}^{\boldsymbol{A}_1}_{B_1}\circ\beta_{t_1}\right)(\Psi)\right\|^2}\\
&\underset{(5)}{=} \quad \frac{\left\|\left(\boldsymbol{E}^{\boldsymbol{A}_2}_{B_2}\circ\hat\beta_{t_2,t_1}\circ\boldsymbol{E}^{\boldsymbol{A}_1}_{B_1}\circ\beta_{t_1}\right)(\Psi)\right\|^2}{\|\Psi\|^2}\\
&\underset{t_1,t_2\to 0}{\longrightarrow} \quad \frac{\left\|\left(\boldsymbol{E}^{\boldsymbol{A}_2}_{B_2}\circ\boldsymbol{E}^{\boldsymbol{A}_1}_{B_1}\right)(\Psi)\right\|^2}{\|\Psi\|^2}\,.
\end{aligned}$$

Unfortunately, an observable $\boldsymbol{E} = \{\boldsymbol{E}_B\}$ is not uniquely characterized by its expectation values, unless it is restricted to be linear. However:

> "It can be maintained that all measurements are reducible to position measurements (pointer readings)." [11]

This suggests that, in principle, the physical interpretation is already completely fixed by the **basic assumption** (4), i.e. by the identification

$$\boldsymbol{E}_B^{\boldsymbol{x}^j} \mathrel{\widehat{=}} \text{multiplication by } \chi_B(x^j)$$

for the observable of the j-component of the position vector.

5 Physical identification of nonlinear observables

A typical measurement procedure for one-particle systems is as follows:

> Apply exterior fields F such that those particles for which $\boldsymbol{A}$ has a value in B asymptotically ($t \to +\infty$) enter the time-dependent region $\mathcal{O}_t \subset \mathbb{R}^3$ while the others are leaving it. Then
>
> $$\mu_B^{\boldsymbol{A}}(\Psi) = \|\Psi\|^{-2} \lim_{t\to+\infty} \left\|\chi_{\mathcal{O}_t}\, \beta_t^F(\Psi)\right\|^2 ,$$
>
> where β_t^F denotes the Schrödinger dynamics adapted to the applied fields.

In this situation $\boldsymbol{E}_B^{\boldsymbol{A}}(\Psi)$ should be identified with the initial wave function that evolves like $\chi_{\mathcal{O}_t}\beta_t^F(\Psi)$ for large t :

$$\lim_{t\to+\infty}\left\|\beta_t^F\left(\boldsymbol{E}_B^{\boldsymbol{A}}(\Psi)\right)-\chi_{\mathcal{O}_t}\beta_t^F(\Psi)\right\|=0\,,$$

i.e.

$$E_B^{\boldsymbol{A}}(\Psi)=\lim_{t\to+\infty}\left(\left(\beta_t^F\right)^{-1}\circ\chi_{\mathcal{O}_t}\circ\beta_t^F\right)(\Psi)\,. \tag{20}$$

As an example let us derive the nonlinear observable of linear momentum for the theory defined by (15)/(13), now for $\vec{x}\in\mathrm{I\!R}^3$ and with the additional restriction $V=0$. Assuming momentum conservation (4) implies[12]

$$\text{probability for } \vec{p}\in\tilde{\mathcal{O}}=\lim_{t\to\infty}\int_{\frac{t}{m}\tilde{\mathcal{O}}}\left|\beta_{D,t}(\Psi)(\vec{x})\right|^2\,\mathrm{d}\vec{x}\,. \tag{21}$$

Therefore, if $\boldsymbol{p}_D^1$ is the observable of linear momentum in 1-direction, $\boldsymbol{E}_B^{\boldsymbol{p}_D^1}(\Psi)$ should coincide with that initial wave function that evolves like $\chi_{\frac{t}{m}B}(\boldsymbol{x}^1)\beta_{D,t}(\Psi)$ for large t :

$$\lim_{t\to+\infty}\left\|\beta_{D,t}\left(\boldsymbol{E}_B^{\boldsymbol{p}_D^1}(\Psi)(\Psi)\right)-\chi_{\frac{t}{m}B}(\boldsymbol{x}^1)\beta_{D,t}(\Psi)\right\|=0\,,$$

i.e.

$$\boldsymbol{E}_B^{\boldsymbol{p}_D^1}(\Psi)=\lim_{t\to+\infty}\left(\beta_{D,-t}\circ\chi_{\frac{t}{m}B}\circ\beta_{D,t}\right)(\Psi)\,. \tag{22}$$

From (21) we easily derive the following.[13]

Lemma 5.1 *For* $\beta_{0,t}=\exp\left(-\frac{i}{\hbar}\Delta t\right)$ *we have*

$$\lim_{t\to+\infty}\beta_{0,-t}\circ\chi_{\frac{t}{m}\tilde{\mathcal{O}}}\circ\beta_{0,t}=\chi_{\tilde{\mathcal{O}}}(\frac{\hbar}{i}\vec{\nabla})$$

in the strong operator topology.

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[12] Actually,

$$\lim_{t\to\infty}\int_{\frac{t}{m}B}\left|\beta_{0,t}(\Psi)(\vec{x})\right|^2\,\mathrm{d}\vec{x}=\int_B\left|\tilde{\Psi}(\vec{p})\right|^2\,\mathrm{d}\vec{p}$$

is well-known to hold for $\beta_{0,t}=\exp\left(-\frac{i}{\hbar}\Delta t\right)$ [7, Sect. 15a].

[13] Note that

$$\lim_{t\to+\infty}\left\|\left(\chi_{\frac{t}{m}B}\circ\beta_{D,t}\circ\chi_{\frac{t}{m}(\mathrm{I\!R}^3\setminus B)}(\frac{\hbar}{i}\vec{\nabla})\right)(\Psi)\right\|=0$$

and

$$\lim_{t\to+\infty}\left\|\left(\chi_{\frac{t}{m}B}\circ\beta_{D,t}\circ\chi_{\frac{t}{m}B}(\frac{\hbar}{i}\vec{\nabla})\right)(\Psi)-\left(\beta_{D,t}\circ\chi_{\frac{t}{m}B}(\frac{\hbar}{i}\vec{\nabla})\right)(\Psi)\right\|=0\,.$$

Since, thanks to

$$\boldsymbol{N}_D \circ \chi_{\frac{t}{m}\tilde{\mathcal{O}}} = \chi_{\frac{t}{m}\tilde{\mathcal{O}}} \circ \boldsymbol{N}_D \,,$$

we have

$$\begin{aligned} \beta_{D,-t} \circ \chi_{\frac{t}{m}\tilde{\mathcal{O}}} \circ \beta_{D,t} &= \boldsymbol{N}_D \circ \beta_{0,-t} \circ \boldsymbol{N}_{-D} \circ \chi_{\frac{t}{m}\tilde{\mathcal{O}}} \circ \boldsymbol{N}_D \circ \beta_{0,t} \circ \boldsymbol{N}_{-D} \\ &= \boldsymbol{N}_D \circ \beta_{0,-t} \circ \chi_{\frac{t}{m}\tilde{\mathcal{O}}} \circ \beta_{0,t} \circ \boldsymbol{N}_{-D} \,, \end{aligned}$$

Lemma 5.1 shows that

$$\boldsymbol{E}_B^{\boldsymbol{p}_D^1}(\Psi) = \text{s-}\lim_{t\to+\infty} \beta_{D,-t} \circ \chi_{\frac{t}{m}B} \circ \beta_{D,t} = \boldsymbol{N}_D \circ \chi_{\frac{t}{m}B}(\frac{\hbar}{i}\partial_1) \circ \boldsymbol{N}_{-D} \,,$$

i.e. the (nonlinear) observable for linear momentum is

$$\boldsymbol{p}_D^j = \left\{ \boldsymbol{N}_D \circ \chi_B(\frac{\hbar}{i}\partial_j) \circ \boldsymbol{N}_{-D} \right\} \quad \text{for } j = 1,2,3 \,.$$

Even though

$$E_{\beta_{D,t}(\Psi)}\left(\vec{\boldsymbol{p}}_D\right) = E_{\beta_{D,t}(\Psi)}\left(\vec{\boldsymbol{p}}_0\right) = \frac{\mathrm{d}}{\mathrm{d}t} \int \vec{x}\, \rho_{\beta_{D,t}(\Psi)}(\vec{x})\, \mathrm{d}\vec{x} \,,$$

$\vec{\boldsymbol{p}}_D$ and $\vec{\boldsymbol{p}}_0$ – contrary to what Weinberg guessed[14] [13, Sec. 2] – do not coincide for $D \neq 0$,

> $\vec{\boldsymbol{p}}_D$ is conserved by $\{\beta_{D,t}\}$ but $\vec{\boldsymbol{p}}_0$ is not.

Finally, note that

$$B_1 \cap B_2 = \emptyset \implies \left\| \boldsymbol{E}_{B_1 \cup B_2}^{\boldsymbol{A}}(\Psi) \right\|^2 = \left\| \boldsymbol{E}_{B_1}^{\boldsymbol{A}}(\Psi) \right\|^2 + \left\| \boldsymbol{E}_{B_2}^{\boldsymbol{A}}(\Psi) \right\|^2 \,,$$

holds for every observable $\boldsymbol{A}$, but:

$$\begin{aligned} B_1 \cap B_2 = \emptyset \quad &\overset{\text{i.g.}}{\not\Longrightarrow} \quad \left\langle \boldsymbol{E}_{B_1}^{\boldsymbol{p}_D^1}(\Psi) \mid \boldsymbol{E}_{B_2}^{\boldsymbol{p}_D^1}(\Psi) \right\rangle = 0 \,, \\ B_1 \cap B_2 = \emptyset \quad &\overset{\text{i.g.}}{\not\Longrightarrow} \quad \boldsymbol{E}_{B_1 \cup B_2}^{\boldsymbol{p}_D^1}(\Psi) = \boldsymbol{E}_{B_1}^{\boldsymbol{p}_D^1}(\Psi) + \boldsymbol{E}_{B_2}^{\boldsymbol{p}_D^1}(\Psi) \,. \end{aligned}$$

This fact is a reasonable consequence of nonlinearity of the dynamics.

[14] The Doebner-Goldin (10) equation can be shown to fit into Weinberg's framework [13] if $c_1 = 1\,, \quad c_2 + 2c_5 = 0\,, \quad c_3 = 0\,, \quad c_4 = -1\,.$

6 Consistent physical interpretation

Let $\mathcal{A}$ be the set of observables for the Schrödinger dynamics $\{\beta_t\}$ and define

$$P_{\mathcal{A}}(\mathcal{H}) \stackrel{\text{def}}{=} \left\{ \boldsymbol{E}_B^{\boldsymbol{A}} : \boldsymbol{A} \in \mathcal{A} \,,\; \mathbb{R} \supset B \text{ Borel} \right\} .$$

Then we expect the following consistency conditions to be fulfilled:

(C1) Let $\boldsymbol{E}, \boldsymbol{E}' \in P_{\mathcal{A}}(\mathcal{H})$. Then

$$\omega_\Psi(\boldsymbol{E}) = \omega_\Psi(\boldsymbol{E}') \;\forall\, \Psi \in \mathcal{H} \setminus \{0\} \implies \boldsymbol{E} = \boldsymbol{E}' .$$

(C2) Enhanced by the semi-ordering $\prec$,

$$\boldsymbol{E}_1 \prec \boldsymbol{E}_2 \stackrel{\text{def}}{\iff} \omega_\Psi(\boldsymbol{E}_1) \leq \omega_\Psi(\boldsymbol{E}_2) \;\forall\, \Psi \in \mathcal{H} \setminus \{0\}$$

and the ortho-complementation $\neg$, thanks to (C1) uniquely characterized by

$$\omega_\Psi\left(\neg \boldsymbol{E}\right) = 1 - \omega_\Psi(\boldsymbol{E}) \quad \text{for } \boldsymbol{E} \in P_{\mathcal{A}}(\mathcal{H}) \,,\; \Psi \in \mathcal{H} \,,$$

$P_{\mathcal{A}}(\mathcal{H})$ becomes a σ-complete orthomodular lattice (**quantum logic**).

(C3) The **pure states**[15] on $\left(P_{\mathcal{A}}(\mathcal{H}), \prec, \neg\right)$ are exactly those of the form ω_Ψ, $\Psi \in \mathcal{H} \setminus \{0\}$.

(C4) The Schrödinger dynamics $\beta_{D,t}$ corresponds to a family of automorphisms α_t of $\left(P_{\mathcal{A}}(\mathcal{H}), \prec, \neg\right)$,

$$\alpha_t(\boldsymbol{E}) = \beta_{D,-t} \circ \boldsymbol{E} \circ \beta_{D,t} \;\text{ for } \boldsymbol{E} \in P_{\mathcal{A}}(\mathcal{H}) .$$

---

[15] A ***state*** on $\left(P_{\mathcal{A}}(\mathcal{H}), \prec, \neg\right)$ is a σ-additive mapping $\omega : P_{\mathcal{A}}(\mathcal{H}) \longrightarrow [0,1]$ with $\omega(\mathbf{1}) = 1$, fulfilling the Jauch-Piron property

$$\forall\, \boldsymbol{E}, \boldsymbol{E}' \in P_{\mathcal{A}}(\mathcal{H}) : \quad \omega(\boldsymbol{E}) = \omega(\boldsymbol{E}') = 1 \Longrightarrow \omega(\boldsymbol{E} \wedge \boldsymbol{E}') = 1 .$$

This guarantees for arbitrary $\boldsymbol{E}, \boldsymbol{E}' \in P_{\mathcal{A}}(\mathcal{H})$ that

$$\omega_\Psi(\boldsymbol{E}) = 1 \iff \omega_\Psi(\boldsymbol{E}') = 1 \quad \forall\, \Psi \in \mathcal{H} \setminus \{0\}$$

implies

$$\omega_\Psi(\boldsymbol{E}) = 0 \iff \omega_\Psi(\boldsymbol{E}') = 0 \quad \forall\, \Psi \in \mathcal{H} \setminus \{0\} \;.$$

Lemma 6.1 *Let $\mathcal{A}$ be the set of observables for the Schrödinger dynamics $\{\beta_t\}$ fulfilling conditions (C1)–(C4). Then:*

(i) There are **no problems with mixed states** *:*

$$\sum_{\nu=0}^{\infty} \lambda_\nu \omega_{\Psi_\nu}(\boldsymbol{E}) = \sum_{\nu'=0}^{\infty} \lambda'_{\nu'} \omega_{\Psi'_{\nu'}}(\boldsymbol{E}) \quad \forall\, \boldsymbol{E} \in P_{\mathcal{A}}(\mathcal{H})$$
$$\Longrightarrow \sum_{\nu=0}^{\infty} \lambda_\nu \omega_{\beta_t(\Psi_\nu)}(\boldsymbol{E}) = \sum_{\nu'=0}^{\infty} \lambda'_{\nu'} \omega_{\beta_t(\Psi'_{\nu'})}(\boldsymbol{E}) \quad \forall\, t \in \mathbb{R}\,,\ \boldsymbol{E} \in P_{\mathcal{A}}(\mathcal{H})\,.$$

(ii) **Primitive causality** *holds in the following sense:*

For every $t \in \mathbb{R}$ and for every $\boldsymbol{E} \in P_{\mathcal{A}}(\mathcal{H})$ there is a $\boldsymbol{E}_t \in P_{\mathcal{A}}(\mathcal{H})$ fulfilling

$$\omega_\Psi(\boldsymbol{E}) = \omega_{\beta_t(\Psi)}(\boldsymbol{E}_t) \quad \forall\, \Psi \in \mathcal{H} \setminus \{0\}\,.$$

(iii) There is a nonlinear ***Heisenberg picture****:*

Let $t \in \mathbb{R}$ and $\boldsymbol{E} \in P_{\mathcal{A}}(\mathcal{H})$. Then[16] *there is a $\boldsymbol{E}(t) \in P_{\mathcal{A}}(\mathcal{H})$ fulfilling*

$$\omega_{\beta_t(\Psi)}(\boldsymbol{E}) = \omega_\Psi\Big(\boldsymbol{E}(t)\Big) \quad \forall\, \Psi \in \mathcal{H}\,.$$

For the nonlinear Schrödinger dynamics given by (15)/(13) a set of GPVM's fulfilling (C1)–(C4) is

$$\mathcal{A} = \{\boldsymbol{A}_D = \{\boldsymbol{N}_D \circ \chi_B(\boldsymbol{A}) \circ \boldsymbol{N}_{-D}\} :\ \boldsymbol{A} \text{ self-adjoint}\}\,. \tag{23}$$

Here $\Big(P_{\mathcal{A}}(\mathcal{H}), \prec, \neg\Big)$ is isomorphic to the standard quantum logic, the isomorphism $\gamma :\ P(\mathcal{H}) \longrightarrow P_{\mathcal{A}}(\mathcal{H})$ being implemented by $\boldsymbol{N}_D$:

$$\gamma(\boldsymbol{P}) = \boldsymbol{N}_D \circ \boldsymbol{P} \circ \boldsymbol{N}_{-D} \ \text{ for } \boldsymbol{P} \in P(\mathcal{H})\,.$$

Since, by the correspondence

	nonlinear theory	linear theory
state vector	$\boldsymbol{N}_D(\Psi)$	Ψ
time-evolution	$\beta_{D,t} = \boldsymbol{N}_D \circ \beta_{0,t} \circ \boldsymbol{N}_{-D}$	$\beta_{0,t} = e^{-\frac{i}{\hbar}\boldsymbol{H}^0_t t}$
selected observables	$\boldsymbol{A}_D = \{\boldsymbol{N}_D \circ \chi_B(\boldsymbol{A}) \circ \boldsymbol{N}_{-D}\}$	$\boldsymbol{A}$
position observable	$\vec{\boldsymbol{x}}_D = \{\chi_B(\vec{\boldsymbol{x}})\}$	$\vec{\boldsymbol{x}} =$ multipl. by $\vec{x}$
expectation values	$E_{\boldsymbol{N}_D(\Psi)}(\boldsymbol{A}_D)$	$E_\Psi(\boldsymbol{A})$

[16] Obviously $\boldsymbol{E}(t) \,\widehat{=}\, \alpha_t(\boldsymbol{E})$:

$$\underbrace{\omega_{\beta_{D,t}(\Psi)}(\boldsymbol{E})}_{\text{Schrödinger picture}} = \underbrace{\omega_\Psi\left(\alpha_t(\boldsymbol{E})\right)}_{\text{Heisenberg picture}} \qquad \forall\, t \in \mathbb{R}\,,\ \Psi \in \mathcal{H} \setminus \{0\}\,,\ \boldsymbol{E} \in P_{\mathcal{A}}(\mathcal{H})\,.$$

our special nonlinear theory becomes physically equivalent to standard linear quantum mechanics, we see:

> A nonlinear theory need not a priori be less consistent than a linear one.

Challenge:

> Find a suitable Schrödinger dynamics $\{\beta_t\}$ (defining the formal singularities) for the original Doebner-Goldin equation (9) and a set $\mathcal{A}$ of GPVM's respecting (C1)–(C4).

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